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NOTE

A NOTE ON A SPANNING 3-TREE

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A tree T is called a k-tree, if the maximum degree of T is at most k. In this paper, we prove that if G is an n-connected graph with independence number at most $n+m+1$ $(n \geq 1, n \geq m \geq 0)$, then G has a spanning 3-tree T with at most m vertices of degree 3.

In this paper, we consider only finite graphs without loops or multiple edges. For standard graph-theoretic terminologies not explained in this paper, we refer the reader to [1]. We denote the degree of a vertex x in a graph G by $d_G(x)$. Let $\alpha(G)$ be the independence number of a graph G.

A tree T is called a k-tree if $d_T(x) \leq k$ for every $x \in V(T)$. For a tree T, let $V_T^k = \{x \in V(T) | d_T(x) = k\}$. Neumann-Lara and Rivera-Campo obtained an independence number condition for a graph to have a spanning k-tree with bounded number of vertices with degree k.

Theorem 1 **(Neumann-Lara and Rivera-Campo [2]).** *Let G be an n-connected graph with $\alpha(G) \leq n(k-2)+m+1$ $(k \geq 4, n \geq m \geq 0)$. Then G has a spanning k-tree T with $|V_T^k| \leq m$.*

The purpose of this paper is to show the case $k=3$ of Theorem 1.

Theorem 2. *Let G be an n-connected graph with $\alpha(G) \leq n+m+1$ $(n \geq 1, n \geq m \geq 0)$. Then G has a spanning 3-tree T with $|V_T^3| \leq m$.*

$K_{n,n+m+2}$ shows that Theorem 2 is best possible.

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We call a system a set of paths (including single vertices) and cycles whose elements are pair-wisely vertex-disjoint. Let $\mathcal{S}$ be a system. For $S\in\mathcal{S}$, we put $f(S)=2$ if S is a path of length (number of edges) at least 2; otherwise $f(S)=1$. For $\mathcal{T}\subseteq\mathcal{S}$, we let $V(\mathcal{T})=\cup_{S\in\mathcal{T}}V(S)$ and $f(\mathcal{T})=\sum_{S\in\mathcal{T}}f(S)$. We call $\mathcal{S}$ a k-ended system if $f(\mathcal{S})\le k$, and in addition, if $V(\mathcal{S})=V(G)$, we call $\mathcal{S}$ a spanning k-ended system.

In [3], Win proved the following theorem.

Theorem 3 (Win [3]). *Let G be an n-connected graph with $\alpha(G)\le n+k-1$ $(n\ge 1, k\ge 2)$. Then G has a spanning k-ended system.*

By Theorem 3, we have only to prove the following proposition.

Proposition 1. *Let G be an n-connected graph. Suppose that G has a spanning k-ended system $\mathcal{S}$ with $2\le k\le n+2$. Then G has a 3-tree T with $|V_T^3|\le k-2$.*

Proof. In case $|\mathcal{S}|=1$, we can obtain a spanning 3-tree T with $|V_T^3|=0$. Hence we obtain the desired conclusion and we may assume $|\mathcal{S}|\ge 2$.

Claim 1. *For each $2\le i\le|\mathcal{S}|$, there exist a 3-tree F and $\mathcal{C}\subseteq\mathcal{S}$ which satisfy the following three properties:*
(i) $V(F)=V(\mathcal{C})$, (ii) $|\mathcal{C}|=i$ and (iii) $|V_F^3|\le f(\mathcal{C})-2$.

Proof. We prove this claim by induction on i. We can easily see the case $i=2$. Suppose that $i\ge 3$ and there exist a 3-tree F' and $\mathcal{C}'\subsetneq\mathcal{S}$ such that $V(F')=V(\mathcal{C}')$, $|\mathcal{C}'|=i-1$ and $|V_{F'}^3|\le f(\mathcal{C}')-2$. Since G is an n-connected graph and F' is a tree with $|V_{F'}^3|\le f(\mathcal{C}')-2<k-2\le n$, there exist $S\in\mathcal{S}-\mathcal{C}'$ and $e\in E(F',S)$ such that $V(e)\cap V_{F'}^3=\emptyset$ (we use $V(e)$ to denote the set of vertices incident with e). If S is a cycle, then let f be an edge of S such that $V(e)\cap V(f)\neq\emptyset$, and define F as $V(F)=V(F')\cup V(S)$ and $E(F)=E(F')\cup\{e\}\cup(E(S)-\{f\})$. The other case, define F as $V(F)=V(F')\cup V(S)$, $E(F)=E(F')\cup\{e\}\cup E(S)$. Then F is a 3-tree which satisfies $V(F)=V(\mathcal{C}'\cup\{S\})$ and $|V_F^3|\le|V_{F'}^3|+f(S)\le f(\mathcal{C}')-2+f(S)=f(\mathcal{C}'\cup\{S\})-2$. ∎

By Claim 1, there exists a spanning 3-tree T which satisfies $|V_T^3|\le f(\mathcal{S})-2\le k-2$. This completes the proof of Proposition 1 and the proof of Theorem 2. ∎

In [3], by using Theorem 3, Win proved that a graph G with $\alpha(G)\le n+k-1$ $(n\ge 1,\ k\ge 2)$ has a spanning tree with at most k ends. On the other hand, since a 3-tree with at most m vertices of degree 3 has at most $m+2$ ends, Theorem 2 shows that a graph G with $\alpha(G)\le n+k-1$ $(2\le k\le n+2)$ has a spanning 3-tree with at most k ends. This strengthens Win's result in the case where $k\le n+2$.

References

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